

Bialgebra deformations and algebras of trees*

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Abstract

Let A denote a bialgebra over a field k and let $A_t = A[[t]]$ denote the ring of formal power series with coefficients in A . Assume that A is also isomorphic to a free, associative algebra over k . We give a simple construction which makes A_t a bialgebra deformation of A . In typical applications, A_t is neither commutative nor cocommutative. In the terminology of [1], A_t is a quantum group. This construction yields quantum groups associated with families of trees.

1 Introduction

Let A denote a bialgebra over a field k and let $A_t = A[[t]]$ denote the ring of formal power series with coefficients in A . Assume that A is also isomorphic to a free, associative algebra over k . We give a simple construction which makes A_t a bialgebra deformation of A . In typical applications, A_t is neither commutative nor cocommutative. In the terminology of [1], A_t is a quantum group. This is an extended abstract. The final detailed version of this paper [6] containing proofs and additional examples has been submitted for publication elsewhere.

An interesting class of examples is obtained by taking the bialgebra A to be the Hopf algebra associated with certain families of trees as in [4]. In fact, these examples are closely related to each other and to algorithms pertaining to differential operators [5].

Formal deformations of bialgebras and quantum groups has also been studied from related viewpoints by Drinfeld [1], Gerstenhaber [2], and Gerstenhaber and Schack [3].

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2 Power series and the completed tensor product

Suppose that k is a field and A and B are k -algebras. We let $A_t = A[[t]]$ denote the ring of formal power series over A with its usual multiplication. Let $f : A \longrightarrow B_t$ be a k -linear map and write $f(a) = \sum_{n=0}^{\infty} c_f(a, n)t^n$ for $a \in A$. Define a k -linear map $\hat{f} : A_t \longrightarrow B_t$ by

$$\hat{f}(\mathbf{a}) = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} c_f(a_i, j) \right) t^n, \quad \text{for } \mathbf{a} = \sum_{n=0}^{\infty} a_n t^n \in A_t.$$

Observe that we can define a category $(Alg_k)_t$, whose objects are A_t and whose morphisms are k -linear maps $f : A_t \longrightarrow B_t$ satisfying $f = \widehat{f|_A}$, where $f|_A$ is the restriction of f to A .

We define the *completed tensor product* $A_t \hat{\otimes}_{k_t} B_t$ of A_t and B_t over k_t in this category to be $(A \otimes_k B)_t$. For $\mathbf{a} = \sum_{n=0}^{\infty} a_n t^n \in A_t$ and $\mathbf{b} = \sum_{n=0}^{\infty} b_n t^n \in B_t$ we let

$$\mathbf{a} \hat{\otimes} \mathbf{b} = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} a_i \otimes b_j \right) t^n.$$

Suppose that $f : A_t \longrightarrow A'_t$ and $g : B_t \longrightarrow B'_t$ are morphisms. We define a morphism of completed tensor products $f \hat{\otimes} g : A_t \hat{\otimes}_{k_t} B_t \longrightarrow A'_t \hat{\otimes}_{k_t} B'_t$ by setting $(f \hat{\otimes} g)|_{A \otimes_k B}(a \otimes b) = f(a) \hat{\otimes} g(b)$ for $a \in A$ and $b \in B$. The usual formalism for the linear tensor product of maps translates to

$$(f \hat{\otimes} g)(\mathbf{a} \hat{\otimes} \mathbf{b}) = f(\mathbf{a}) \hat{\otimes} g(\mathbf{b})$$

for $\mathbf{a} \in A_t$ and $\mathbf{b} \in B_t$ in this category. Note that if f and g are also algebra maps, then the morphism $f \hat{\otimes} g : A_t \hat{\otimes}_{k_t} B_t \longrightarrow A'_t \hat{\otimes}_{k_t} B'_t$ is an algebra map.

Now let C be any coalgebra over k . A sequence of elements $c_0, c_1, c_2, \dots \in C$ is called a sequence of divided powers if

$$\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j \quad \text{and} \quad \epsilon(c_n) = \delta_{0,n} \quad \text{for all } n \geq 0.$$

Probably the most basic example of a sequence of divided powers arises from a primitive element in a bialgebra. Suppose that A is a bialgebra over k and that $\ell \in A$ is primitive. Since Δ is multiplicative, we calculate by the binomial theorem that $\Delta(\ell^n) = (\Delta(\ell))^n = (\ell \otimes 1 + 1 \otimes \ell)^n = \sum_{i=0}^n \binom{n}{i} (\ell^{n-i} \otimes \ell^i)$ for $n \geq 0$. Thus c_0, c_1, c_2, \dots is a sequence of divided powers, where $c_n = \frac{\ell^n}{n!}$ for $n \geq 0$.

3 Deformations of certain enveloping algebras

The notions of algebra, coalgebra, bialgebra and Hopf algebra in the category $(Alg_k)_t$ are the same as those in the category of vector spaces over the field k except, of course, the structure maps are required to be morphisms. Let (A, m, η) be an algebra over k , where $m : A \otimes A \longrightarrow A$ is multiplication and $\eta : k \longrightarrow A$ defines the unity of A . Then $(A_t, \hat{m}, \hat{\eta})$ is an algebra in $(Alg_k)_t$. It is easy to see that $\hat{m}(a \hat{\otimes} b) = ab$ for $a, b \in A_t$. A morphism $f : A_t \longrightarrow B_t$ is a morphism of algebras if and only if $f|_A : A \longrightarrow B_t$ is a map of k -algebras.

The proof of the proposition below is really a matter of unravelling definitions.

Proposition 1 *Suppose that A is an algebra over a field k with a k -coalgebra structure (A, Δ, ϵ) . Then $(A_t, \hat{\Delta}, \hat{\epsilon})$ is a coalgebra in $(Alg_k)_t$. Furthermore*

$$\hat{\Delta}(a) = \sum_{n=0}^{\infty} (\Delta(a_n)) t^n \quad \text{and} \quad \hat{\epsilon}(a) = \sum_{n=0}^{\infty} \epsilon(a_n) t^n$$

for $a = \sum_{n=0}^{\infty} a_n t^n \in A_t$.

Suppose that (A_t, Δ, ϵ) is a coalgebra in $(Alg_k)_t$. We say that $K \in A_t$ is grouplike if

$$\Delta(K) = K \hat{\otimes} K \quad \text{and} \quad \epsilon(K) = 1.$$

We say that $\ell \in A_t$ is nearly primitive if

$$\Delta(\ell) = \ell \hat{\otimes} K + H \hat{\otimes} \ell$$

for some grouplike elements $K, H \in A_t$. If $K = H = 1$ then ℓ is said to be primitive.

For an algebra A over a field k of characteristic 0 we let $\exp(at) = \sum_{n=0}^{\infty} (\frac{a^n}{n!}) t^n \in A_t$. The following corollary gives the relationship between sequences of divided powers and grouplike elements.

Corollary 1 *Suppose that A is an algebra over a field k which has a k -coalgebra structure (A, Δ, ϵ) . Let $(A_t, \hat{\Delta}, \hat{\epsilon})$ be the resulting coalgebra in $(Alg_k)_t$. Then:*

- (a) *Let $K = \sum_{n=0}^{\infty} a_n t^n \in A_t$. Then K is grouplike if and only if a_0, a_1, a_2, \dots is a sequence of divided powers in A .*
- (b) *Suppose that the characteristic of k is 0. If $a \in A$ is primitive, then $K = \exp(at)$ is a grouplike element of A_t .*

Now we construct deformations of enveloping algebras over a field of characteristic 0 which are free as associative algebras on a space of primitives.

Theorem 1 *Suppose that V is a vector space over a field k of characteristic 0. Turn the tensor algebra (A, m, η) of V into a bialgebra $(A, m, \eta, \Delta, \epsilon)$ by defining $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ and $\epsilon(\ell) = 0$ for $\ell \in V$. Let $p, q \in V$ and write V as a direct sum of subspaces $V = P \oplus P'$, where $P = \text{span}(p, q)$. Then there is bialgebra deformation $(A_t, \hat{m}, \hat{\eta}, \hat{\Delta}, \hat{\epsilon})$ of $(A, m, \eta, \Delta, \epsilon)$ such that*

- a) $\hat{\Delta}(\ell) = \ell \hat{\otimes} 1 + 1 \hat{\otimes} \ell$ for $\ell \in P$,
- b) $K = \exp(pt)$ and $H = \exp(qt)$ are grouplike elements of $(A_t, \hat{\Delta}, \hat{\epsilon})$,
and
- c) $\hat{\Delta}(\ell) = \ell \hat{\otimes} K + H \hat{\otimes} \ell$ for $\ell \in P'$.

We comment that $K = \exp(tp) = 1$ when $p = 0$. If $p \neq 0$ and $q = 0$, for example, then the deformation of the theorem is not cocommutative. If $\dim(V) > 1$ then the free algebra A is not commutative. In this case the deformation of the theorem is not commutative.

4 Deformations of bialgebras trees

In this section we give an example from [4] involving bialgebras of trees to which Theorem 1 applies. Let k be a field of characteristic 0. Let \mathcal{T} be the set of finite rooted trees, and let $k\{\mathcal{T}\}$ be the k -vector space which has \mathcal{T} as a basis.

We first define a coalgebra structure on $k\{\mathcal{T}\}$. If $t \in \mathcal{T}$ is a tree whose root has children s_1, \dots, s_r , the coproduct $\Delta(t)$ is the sum of the 2^r terms $t_1 \otimes t_2$, where the children of the root of t_1 and the children of the root of t_2 range over all 2^r possible partitions of the children of the root of t into two subsets. The map ϵ which sends the trivial tree to 1 and every other tree to 0 is a counit for this coproduct. It is easy to see that comultiplication is cocommutative.

We now define an algebra structure on $k\{\mathcal{T}\}$. Suppose that $t_1, t_2 \in \mathcal{T}$ are trees. Let s_1, \dots, s_r be the children of the root of t_1 . If t_2 has $n + 1$ nodes (counting the root), there are $(n + 1)^r$ ways to attach the r subtrees of t_1 which have s_1, \dots, s_r as roots to the tree t_2 by making each s_i the child of some node of t_2 . The product $t_1 t_2$ is defined to be the sum of these $(n + 1)^r$ trees. It turns out that this product is associative, and that the trivial tree e consisting only of the root is a right and left unit. It can also be shown

that the maps defining the coalgebra structure are algebra homomorphisms, so that $A = k\{T\}$ is a bialgebra.

For technical reasons, we require that nodes of the tree be ordered. We say that a rooted finite tree is *ordered* in case there is a partial ordering on the nodes such that the children of each node are non-decreasing with respect to the ordering. To define the product of two ordered trees t_1 and t_2 , compute the product of the underlying trees, and order the nodes in the product so that the nodes which originally belonged to each tree retain the same relative order to each other, but all the nodes that originally belonged to t_1 are greater in the ordering than the nodes that originally belonged to t_2 .

Let X be the set of trees whose root has one child. Then $A \cong k\langle I(X) \rangle$ as an algebra [4]. Applying Theorem 1 yields a bialgebra deformation A_t of A which is neither commutative nor cocommutative.

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